

Modelli 1 @ Clamfim

Massimi e minimi

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Exercise Find minimum of $f(x, y, z) = x^2 + 2y^2 + 2yz + 3z^2$ subject to constraints $x + y + z = 1$ and $2x + y + 3z = \mu$

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Lagrangian:

$$L(x, y, z; m, n) = x^2 + 2y^2 + 2yz + 3z^2 - m(x + y + z - 1) - n(2x + y + 3z - \mu)$$

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Critical point equations

$$\begin{cases} 2x = m + 2n \\ 4y + 2z = m + n \\ 2y + 6z = m + 3n \\ x + y + z = 1 \\ 2x + y + 3z = \mu \end{cases}$$

Get x, y, z from the first 3 eqs in terms of the multipliers n, m

$$\begin{cases} x = \frac{m + 2n}{2} \\ y = \frac{m}{5} \\ z = \frac{m + 5n}{10} \\ x + y + z = 1 \\ 2x + y + 3z = \mu \end{cases}$$

So, putting x, y, z in the last two eqs

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Sufficient conditions: Bordered Hessian

Find max or min of $f(x, y)$ under the constraint $g(x, y) = 0$

Lagrangian $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

After solving the system

$$\begin{cases} f'_x(x, y) - \lambda g'_x(x, y) = 0, \\ f'_y(x, y) - \lambda g'_y(x, y) = 0, \\ g(x, y) = 0. \end{cases}$$

evaluate

$$\Lambda = \det \begin{bmatrix} L''_{xx} & L''_{xy} & g_x \\ L''_{xy} & L''_{yy} & g_y \\ g_x & g_y & 0 \end{bmatrix}$$

$\Lambda > 0$ maximum

$\Lambda < 0$ minimum

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Critical point system

$$\begin{cases} 2(x - 2) - 2mx = 0 \\ 2(y - 2) - 2my = 0 \\ x^2 + y^2 = 1 \end{cases} \implies \begin{cases} (1 - m)x = 2 \\ (1 - m)y = 2 \\ x^2 + y^2 = 1 \end{cases}$$

Solving we get

$$\left(\frac{2}{1-m}\right)^2 + \left(\frac{2}{1-m}\right)^2 = 1$$

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$$\text{Bordered Hessian } \Lambda = \begin{bmatrix} 2 - 2m & 0 & 2x \\ 0 & 2 - 2m & 2y \\ 2x & 2y & 0 \end{bmatrix}$$

For $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$, $m = 1 - 2\sqrt{2}$ we then get

$$\Lambda = \begin{bmatrix} 4\sqrt{2} & 0 & \sqrt{2} \\ 0 & 4\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$$

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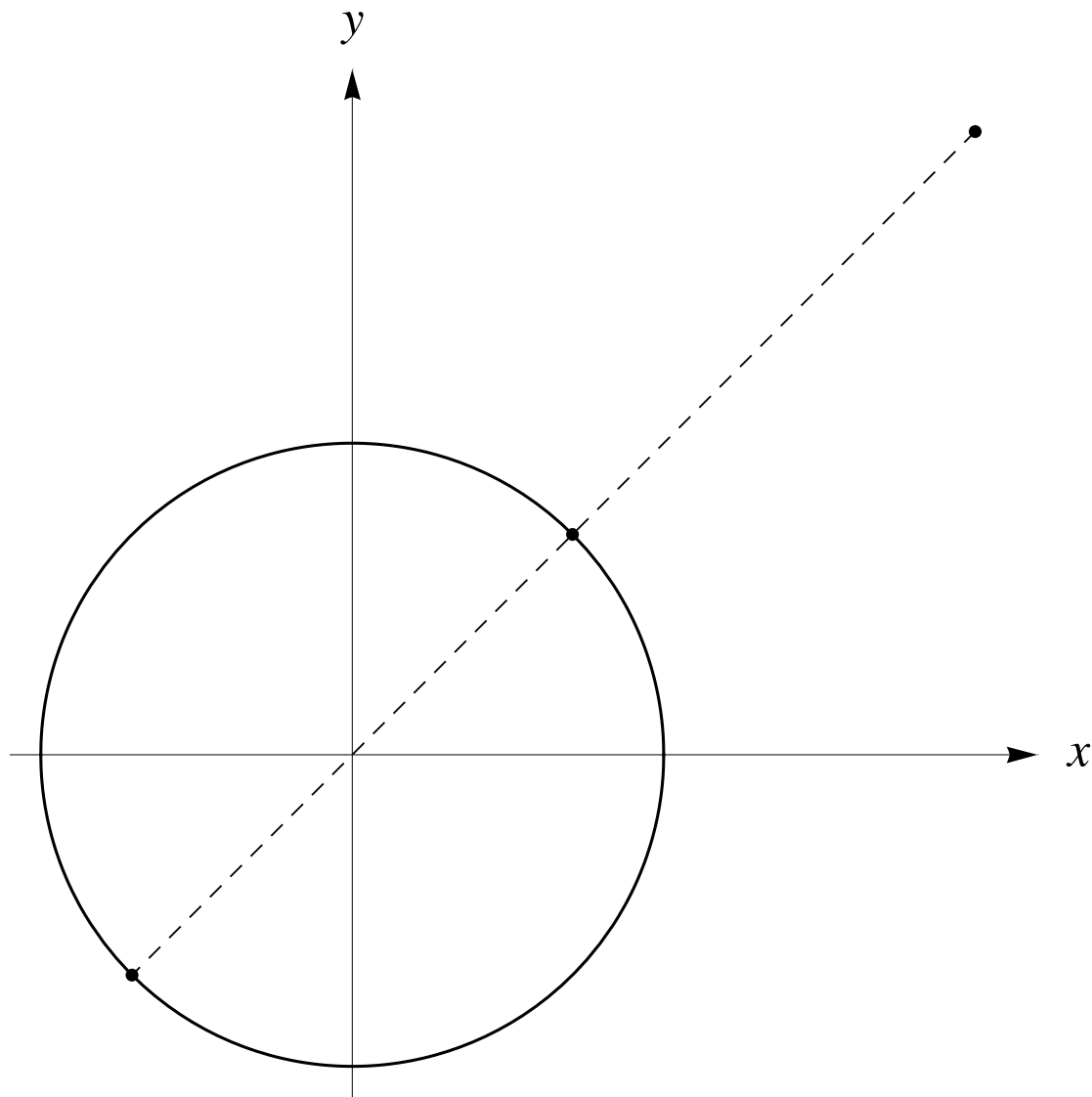
$$\Lambda = \begin{bmatrix} 4\sqrt{2} & 0 & \sqrt{2} \\ 0 & 4\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$$

So that $\det \Lambda = -16\sqrt{2}$ minimum

The other critical point give rise to

$$\Lambda = \begin{bmatrix} -4\sqrt{2} & 0 & -\sqrt{2} \\ 0 & -4\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 0 \end{bmatrix}$$

here $\det \Lambda = 16\sqrt{2}$ maximum



Study maxima and minima of $f(x, y) = 2x + y$ subject to constraint $x^{1/4}y^{3/4} = 1, \quad x > 0, y > 0$

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Critical point equations

$$\begin{cases} 2 - \frac{my^{3/4}}{4x^{3/4}} = 0 \\ 1 - \frac{3mx^{1/4}}{4y^{1/4}} = 0 \\ x^{1/4}y^{3/4} = 1 \end{cases}$$

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Divide the first two equations obtaining

$$\begin{cases} 6 = \frac{y}{x} \\ x^{1/4}y^{3/4} = 1 \end{cases}$$

Conclusion $x = 6^{-3/4}$, $y = 6^{1/4}$, $m = \frac{4 \times 2^{1/4}}{3^{3/4}}$

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Bordered Hessian

$$\Lambda = \begin{bmatrix} \frac{3my^{3/4}}{16x^{7/4}} & -\frac{3m}{16x^{3/4}y^{1/4}} & \frac{y^{3/4}}{4x^{3/4}} \\ -\frac{3m}{16x^{3/4}y^{1/4}} & \frac{3mx^{1/4}}{16y^{5/4}} & \frac{3x^{1/4}}{4y^{1/4}} \\ \frac{y^{3/4}}{4x^{3/4}} & \frac{3x^{1/4}}{4y^{1/4}} & 0 \end{bmatrix}$$

Substituting critical point

$$\Lambda = \begin{bmatrix} \frac{3 \times 3^{3/4}}{\sqrt[4]{2}} & -\frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{3^{3/4}}{2\sqrt[4]{2}} \\ -\frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{1}{4\sqrt[4]{6}} & \frac{3^{3/4}}{4\sqrt[4]{2}} \\ \frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{3^{3/4}}{4\sqrt[4]{2}} & 0 \end{bmatrix}$$

Substituting critical point

$$\Lambda = \begin{bmatrix} \frac{3 \times 3^{3/4}}{\sqrt[4]{2}} & -\frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{3^{3/4}}{2\sqrt[4]{2}} \\ -\frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{1}{4\sqrt[4]{6}} & \frac{3^{3/4}}{4\sqrt[4]{2}} \\ \frac{3^{3/4}}{2\sqrt[4]{2}} & \frac{3^{3/4}}{4\sqrt[4]{2}} & 0 \end{bmatrix}$$

Then $\det \Lambda = -\frac{3\sqrt[4]{3}}{2^{3/4}}$ thus we have a minimum

Cobb Douglas

$$\begin{aligned} f(x, y) &= x^a y^{1-a} \rightarrow \max \\ \text{sub } px + qy - c &= 0 \end{aligned}$$

Assumptions $0 < a < 1, p, q, c > 0$.

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Assumptions $0 < a < 1, p, q, c > 0$.

Lagrangian $L(x, y; m) = f(x, y) - mw(x, y)$ where $w(x, y) = px + qy - c$. Critical point equations

$$L_x(x, y; m) = ax^{a-1}y^{1-a} - mp = 0 \quad (1a)$$

$$L_y(x, y; m) = (1-a)x^a y^{-a} - mq = 0 \quad (1b)$$

$$L_m(x, y; m) = px + qy - c = 0 \quad (1c)$$

Eliminating m between (1a) and (1b) we get

$$(1 - a)px - aqy = 0 \quad (2a)$$

$$px + qy - c = 0 \quad (2b)$$

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Solving:

$$\begin{cases} x = \frac{ac}{p} \\ y = \frac{c(1 - a)}{q} \\ m = (1 - a)^{1-a} a^a p^{-a} q^{a-1} \end{cases}$$

The critical point is a maximum, in fact the bordered hessian is

$$\begin{bmatrix} (a-1)a \left(\frac{ac}{p}\right)^{a-2} \left(\frac{c-ac}{q}\right)^{1-a} & (1-a)a \left(\frac{ac}{p}\right)^{a-1} \left(\frac{c-ac}{q}\right)^{-a} & p \\ (1-a)a \left(\frac{ac}{p}\right)^{a-1} \left(\frac{c-ac}{q}\right)^{-a} & -\frac{a \left(\frac{ac}{p}\right)^a \left(\frac{c-ac}{q}\right)^{-a} q}{c} & q \\ p & q & 0 \end{bmatrix}$$

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so that

$$\det = \frac{a^{a-1} p^{2-a} q^{a+1}}{c(1-a)^a} > 0$$

Determinare la minima distanza dall'origine del luogo dei punti di equazione $w(x, y) = 0$

1. $w(x, y) = 3x + 5y - 7$

2. $w(x, y) = y - x^2 + 2x - \frac{17}{9}$ ^a

3. $w(x, y) = x^2 + 8xy + 7y^2 - 225$

4. $w(x, y) = x^2 + xy + 3y^2 - 36$

$$^a 9m^3 + 2m^2 - 23m - 34 = (m - 2)(9m^2 + 20m + 17)$$

FACOLTATIVO: Lagrange multipliers: $n = 2$, $m = 1$

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Implicit function theorem: Ulisse Dini 1878

Let Ω an open set in \mathbb{R}^2 and $f : \Omega \rightarrow \mathbb{R}$ a \mathcal{C}^1 function. Suppose there exists $(x_0, y_0) \in \Omega$ such that $f(x_0, y_0) = 0$, $f_y(x_0, y_0) \neq 0$, then there exist $\delta, \varepsilon > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta)$ there is a unique $y = \varphi(x) \in (y_0 - \varepsilon, y_0 + \varepsilon)$ such that:

$$f(x, y) = 0$$

Function $y = \varphi(x)$ is \mathcal{C}^1 in $(x_0 - \delta, x_0 + \delta)$ and for any $x \in (x_0 - \delta, x_0 + \delta)$

$$\varphi'(x) = -\frac{f_x(x, \varphi(x))}{f_y(x, \varphi(x))}$$

PROOF. Assume $f(x_0, y_0) > 0$. Since $f_y(x, y)$ is continuous there is a ball $B_{\delta_1}(x_0, y_0)$ such that $(x, y) \in B_{\delta_1}(x_0, y_0) \implies f_y(x, y) > 0$

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In particular $y \mapsto f(x_0, y)$ is increasing and since we assumed $f(x_0, y_0)$ this implies

$$f(x_0, y_0 + \varepsilon) > 0 \quad \text{and} \quad f(x_0, y_0 - \varepsilon) < 0$$

for ε small enough.

PROOF. Assume $f(x_0, y_0) > 0$. Since $f_y(x, y)$ is continuous there is a ball $B_{\delta_1}(x_0, y_0)$ such that $(x, y) \in B_{\delta_1}(x_0, y_0) \implies f_y(x, y) > 0$

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for ε small enough. Using again continuity of f and, if appropriate narrowing again δ we infer that for any $x \in (x_0 - \delta, x_0 + \delta)$

$$f(x, y_0 + \varepsilon) > 0 \quad \text{and} \quad f(x, y_0 - \varepsilon) < 0$$

In conclusion using continuity of $y \mapsto f(x, y)$ from Bolzano theorem (existence of zeros) we have shown that for any $x \in (x_0 - \delta, x_0 + \delta)$ there is a unique $y = \varphi(x) \in (y_0 - \varepsilon, y_0 + \varepsilon)$ such that

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To prove the second half of the theorem, we have first to show that $\varphi(x)$ is differentiable.

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Recall the Mean value theorem for function of two variables.

Theorem. If $f(x, y)$ is a \mathcal{C}^1 function defined on an open set $V \subset \mathbb{R}^2$ then for any $(x, y) \in V$ if $h, k \in \mathbb{R}$ are such that $(x + h, y + k) \in V$ then

$$f(x + h, y + k) - f(x, y) = \nabla f(\bar{x}, \bar{y}) \cdot (h, k) = f_x(\bar{x}, \bar{y})h + f_y(\bar{x}, \bar{y})k$$

where \bar{x} lies between $x + h$ and x , \bar{y} lies between $y + k$ and y

We use the mean value theorem to proof the implicit function theorem.
Take $h \in \mathbb{R}$ such that $x + h \in (x_0 - \delta, x_0 + \delta)$ in such a way

$$f(x + h, \varphi(x + h)) = 0 = f(x, \varphi(x))$$

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$$\begin{aligned} 0 &= f(x + h, \varphi(x + h)) - f(x, \varphi(x)) \\ &= f_x(x, \varphi(x))h + f_y(x, \varphi(x))(\varphi(x + h) - \varphi(x)) \end{aligned}$$

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thus

$$\frac{\varphi(x + h) - \varphi(x)}{h} = -\frac{f_x(x, \varphi(x))}{f_y(x, \varphi(x))}$$

thesis follow taking limit for $h \rightarrow 0$

Multipliers Theorem. Let $f : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ a \mathcal{C}^1 function. Consider the subset of A

$$M = \{(x, y) \in A : g(x, y) = 0\}$$

being $g : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ a \mathcal{C}^1 function such that $\nabla g(x, y) \neq 0$ for any $(x, y) \in M$. If $(x_0, y_0) \in M$ is a maximum or a minimum for $f(x, y)$ for $(x, y) \in M$ then there is $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

PROOF. Since $\nabla g(x_0, y_0) \neq 0$ we can assume that $g_y(x_0, y_0) \neq 0$, thus from the implicit function theorem there exist $\varepsilon, \delta > 0$ such that for $x \in (x_0 - \delta, x_0 + \delta)$, $y \in (y_0 - \varepsilon, y_0 + \varepsilon)$ we have

$$g(x, y) = g(x, \varphi(x)) = 0$$

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Consider function $x \mapsto f(x, \varphi(x)) := h(x)$ for $x \in (x_0 - \delta, x_0 + \delta)$.

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$$0 = h'(x_0) = f_x(x_0, \varphi(x_0)) + f_y(x_0, \varphi(x_0))\varphi'(x_0) \quad (\text{a})$$

Now use the implicit function theorem which gives

$$\varphi'(x_0) = -\frac{g_x(x_0, \varphi(x_0))}{g_y(x_0, \varphi(x_0))}$$

substituting in (a) recalling that $\varphi(x_0) = y_0$ we get

$$f_x(x_0, y_0)g_y(x_0, y_0) - f_y(x_0, y_0)g_x(x_0, y_0) = 0 \quad (\text{b})$$

Now use the implicit function theorem which gives

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substituting in (a) recalling that $\varphi(x_0) = y_0$ we get

$$f_x(x_0, y_0)g_y(x_0, y_0) - f_y(x_0, y_0)g_x(x_0, y_0) = 0 \quad (\text{b})$$

Now write (b) as

$$\det \begin{vmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{vmatrix} = 0 \quad (\text{c})$$

Since the (c) determinant is zero it follows that its rows are proportional this implies that there exists $\lambda \in \mathbb{R}$ such that

$$(f_x(x_0, y_0), f_y(x_0, y_0)) = \lambda (g_x(x_0, y_0), g_y(x_0, y_0))$$